Inequality Exercise

1. Prove, with the exceptions stated in each case, that the following inequalities are generally true.

[The letters represent positive quantities.]

(a) $m^3 + 1 > m^2 + m$, unless $m = 1$.

(b) $x^5 + y^5 > x^4 y + xy^4$, unless $x = y$.

(c) $(b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2 > bc + ca + ab$, unless $a = b = c$.

(d) $(a + b)(b + c)(c + a) > 8abc$, unless $a = b = c$

(e) $a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 > 6abc$, unless $a = b = c$.

(f) $\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} > 6$, unless $a = b = c$.

(g) $\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{a}{x} + \frac{b}{y}\right) > 4$, unless $\frac{x}{a} = \frac{y}{b}$.

(h) $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > 9$, unless $a = b = c$.

(i) $\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right)\left(\frac{a}{l} + \frac{b}{m} + \frac{c}{n}\right) > 9$, unless $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

(j) $x^7 + y^7 > x^4y^3 + x^3y^4$, unless $x = y$.

(k) $x^7 + 1 > x^6 + x$, unless $x = 1$.

(l) $x^5 + x^{-5} > x^2 + x^{-2}$, unless $x = 1$.

(m) $(a + b + c)^3 > 27abc$, $9(a^3 + b^3 + c^3) > 27abc$, unless $a = b = c$.

(n) $xyz > (x + y - z)(y + z - x)(z + x - y)$, unless $x = y = z$.

2. Show that: $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{1}{2}$, where $n \in \mathbf{N} \backslash \{1\}$.

3. Let n and p be positive integers. Prove that:

$$\frac{1}{n+1} - \frac{1}{n+p+1} < \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(n+p)^2} < \frac{1}{n} - \frac{1}{n+p}.$$

4. Prove that: $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2$.

5. Prove that: $\frac{1}{2\sqrt{s}} < \frac{1}{4^s} C^s_{2s} < \frac{1}{\sqrt{2s+1}}$

6. Prove that: $\cot\frac{\theta}{2} \ge 1 + \cot\theta$. $(0 < \theta < \pi)$

7. Prove that: $\sqrt{(a+c)(b+d)} \ge \sqrt{ab} + \sqrt{cd}$ (a, b, c, d > 0)

8. Prove that: $\left(\frac{a^3 + b^3}{2}\right) \ge \left(\frac{a+b}{2}\right)^3$ (a, b > 0)

9. Let $a_1, a_2, \dots, a_n$ form an arithmetic progression $(a_i > 0)$.

Prove that: $\sqrt{a_1 a_n} \le \sqrt[n]{a_1 a_2 \dots a_n} \le \frac{a_1 + a_n}{2}$. In particular $\sqrt{n} < \sqrt[n]{n!} < \frac{n+1}{2}$.

10. Prove the Cauchy-Bunyakovskii-Schwarz (CBS) inequality:

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1{}^2 + a_2{}^2 + ... + a_n{}^2)(b_1{}^2 + b_2{}^2 + ... + b_n{}^2).$$

11. Prove that: $a_1 + a_2 + ... + a_n \leq \sqrt{n(a_1{}^2 + a_2{}^2 + ... + a_n{}^2)}$.

12. Prove that: $(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \geq n^2$.

13. Prove that: $\sqrt[3]{3} > \sqrt[4]{4} > \sqrt[5]{5} > .... > \sqrt[n]{n} > \sqrt[n+1]{n+1} > ....$

14. Prove that: $2 > \sqrt{3} > \sqrt[3]{4} > \sqrt[4]{5} > .... > \sqrt[n-1]{n} > \sqrt[n]{n+1} > ....$

15. Let a > 1 and n be a positive integer. Prove that: $a^n - 1 \geq n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right)$.

16. Show that: $(n!)^2 > n^n$ and $2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n) < (n + 1)^n$.

17. Show that: $(x + y + z)^3 > 27xyz$.

18. Show that: $n^n > 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1)$.

19. If n is a positive integer greater than 1 , show that: $2^n > 1 + n\sqrt{2^{n-1}}$.

20. Show that: $(n!)^3 < n^n\left(\frac{n+1}{2}\right)^{2n}$.

21. If x, y, z are lengths of the three sides of a triangle, show that:

(i) $(x + y + z)^3 > 27(x + y - z)(y + z - x)(z + x - y)$

(ii) $xyz > (x + y - z)(y + z - x)(z + x - y)$

22. Show that : $1!\, 3!\, 5! \ldots (2n - 1)! > (n!)^n$.

23. If a, b, c denote the sides of a triangle, show that:

(i) $a^2 (p - q)(p - r) + b^2 (q - r)(q - p) + c^2 (r - p)(r - q) \geq 0$, where $p, q, r \in \mathbf{R}$.

(ii) $a^2yz + b^2zx + c^2xy \leq 0$, if $x + y + z = 0$.

24. Show that if a, b, c are all positive, the expressions:

$$a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b)$$

and $$a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b)$$

are both positive.

25. Show that: $(a + b + c + d) (a^3 + b^3 + c^3 + d^3) \geq (a^2 + b^2 + c^2 + d^2)^2$.

26. If x is positive, show that $\frac{x}{1+x} < \ln(1+x) < x$.

27. If n is a positive integer and x < 1, show that: $\frac{1-x^{n+1}}{n+1} < \frac{1-x^n}{n}$.

28. Prove that: $\frac{1}{m}\log(1+a^m) < \frac{1}{n}\log(1+a^n)$, where $m > n > 0$ and $a > 0$.

29. **(a)** Prove that $\ln\left(\frac{1+y}{1-y}\right) < \frac{1}{1-y} - \frac{1}{1+y}$, where $0 < y < 1$.

(b) By putting $y = \frac{c}{x}$ where $x > c > 0$ in (a) , or otherwise, show that: $\left(\frac{a+c}{a-c}\right)^a < \left(\frac{b+c}{b-c}\right)^b$,

where a, b, c are in descending order of magnitude.

30. If α, β are positive quantities, and $\alpha > \beta$, show that: $\left(1+\frac{1}{\alpha}\right)^\alpha > \left(1+\frac{1}{\beta}\right)^\beta$.

Hence show that if $n > 1$, the value of $\left(1+\frac{1}{n}\right)^n$ lies between 2 and 2.718….

31. Show that the sum of the m^{th} powers of the first n even numbers is greater than: $n(n + 1)^m$, if $m \in \mathbb{N} / \{1\}$.

32. Show that: $27(a^4 + b^4 + c^4) > (a + b + c)^4$.

33. Show that: $n(n + 1)^3 \le 8(1^3 + 2^3 + 3^3 + \ldots + n^3)$.

34. Show that: $(x_1 + x_2 + x_3)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) \ge 9$.

35. Show that: **(i)** $e^x > 1 + x$ for $x \ne 0$.

(ii) $x + \frac{x^3}{3} < \tan x, \quad x \in \left(0, \frac{\pi}{2}\right)$.

36. If $b^2 < ac$ and $a + 2b + c > 0$, show that: $4a + 4b + c > 0$.

37. Prove that: $C^n_k \frac{1}{n^k} \le \frac{1}{k!}$ and $\left(1+\frac{1}{n}\right)^n < 3$.

38. **(i)** Prove the inequality $2(x^2 + y^2) \ge (x + y)^2$, where x, y are real numbers.

When does equality hold?

(ii) If a, b are positive numbers such that $a + b = 1$, prove that: $\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \ge \frac{25}{2}$.

39. **(i)** If $x > 0$, show that $x^3 - x^2 \ge \frac{1}{x^2} - \frac{1}{x^3}$. When does the equality sign hold?

(ii) The 3 sides of a triangle form a Geometric Progression with common ratio r , prove that:

$$\frac{\sqrt{5}-1}{2} < r < \frac{\sqrt{5}+1}{2}.$$

40. Solve for x: $\frac{3x+5}{x-1} > \frac{2x+7}{x-1}$.

41. Let $A + B + C = \pi$, prove that $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \ge 1$.

42. Show that : $a^b b^a < \left(\frac{a+b}{2}\right)^{a+b}$, where $a, b \in \mathbb{N}$.